EE359, Wireless Communications, Winter 2020
Homework 5 (100 pts)

Due: Tuesday February 18, 9am

- Please refer to the homework page on the website (ee359.stanford.edu/homework) for guidelines.
- Late submissions will not be accepted as homework solutions will be released immediately after the deadline.

1. (10 pts) **Outage probability and average probability of error** (Problems 6-10, 6-11):
   (a) (5 pts) Consider a cellular system at 900 MHz with a transmission rate of 64 kbps and multipath fading. Explain which performance metric – average probability of error or outage probability – is more appropriate (and why) for user speeds of 1 mph, 10 mph, and 100 mph.
   (b) (5 pts) Derive the expression for the moment generating function for SNR in Rayleigh fading.

2. (15 pts) **QoS in a cell** (Problem 6-17): Consider a cellular system with circular cells of radius 100 meters. Assume that propagation follows the simplified path-loss model with $K = 1, d_0 = 1$m, and $\gamma = 3$. Assume the signal experiences (in addition to path loss) log-normal shadowing with $\sigma_{\psi_{dB}} = 4$ as well as Rayleigh fading. The transmit power at the base station is $P_t = 100$ mW, the system bandwidth is B = 30 kHz, and the noise PSD $N_0/2$ has $N_0 = 10^{-14}$ W/ Hz. Assuming BPSK modulation, we want to find the cell coverage area (percentage of locations in the cell) where users have average P_b of less than 10^{-4}.
   (a) (4 pts) Find the received power due to path loss at the cell boundary.
   (b) (5 pts) Find the minimum average received power (due to path loss and shadowing) such that, with Rayleigh fading about this average, a BPSK modulated signal with this average received power at a given cell location has $\bar{P}_b < 10^{-4}$.
   (c) (6 pts) Given the propagation model for this system (simplified path loss, shadowing, and Rayleigh fading), find the percentage of locations in the cell where $\bar{P}_b < 10^{-4}$ under BPSK modulation.

3. (10 pts) **Error floors** (Problem 6-21): Consider a wireless channel with average delay spread of 100 ns and a Doppler spread of 80 Hz. Given the error floors due to Doppler and ISI and assuming DPSK modulation in Rayleigh fading and uniform scattering, approximately what range of data rates can be transmitted over this channel with a BER of less than 10^{-4}? Sections 6.4 and 6.5 of the textbook has relevant content on error floors, equation (6.93) is especially useful.

4. (10 pts) **Fading with diversity combining**: Find the outage probability of QPSK modulation at $P_s = 10^{-3}$ for a Rayleigh fading channel with SC diversity for $M = 1$ (no diversity), $M = 2$, and $M = 3$. Assume branch SNRs of $\bar{\gamma}_2 = 10$ dB, $\bar{\gamma}_2 = 15$ dB, and $\bar{\gamma}_3 = 20$ dB. Assume that for $M = 1$, only $\bar{\gamma}_1$ is used, and for $M = 2$, only $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are used.

5. (15 pts) **MRC**:

   (a) (5 pts) Show that the weights a_i maximizing γ_Σ under receiver diversity with MRC are $a_i^2 = r_i^2/N_0B$ for $N_0B/2$ the common noise PSD on each branch. Also show that, with these weights, $\gamma_\Sigma = \sum_i \gamma_i$, as stated in (7.24) of the reader.

   (b) (10 pts) Consider a fading distribution $p(\gamma)$ where $\int_0^\infty p(\gamma)e^{-x\gamma}\,\mathrm{d}\gamma = (1+2\bar{\gamma}x)^{-1/2} \quad \forall x > -1/2\bar{\gamma}$. Find the average P_b for a BPSK modulated signal where the receiver has 2-branch diversity with MRC combining, and each branch has an average SNR of 10 dB and experiences independent fading with distribution $p(\gamma)$.

6. (15 pts) **General diversity scheme** Consider a diversity method which combines the low-complexity of SC with the performance benefits of MRC. The diversity combiner has N receiver branches with independent fading on each arm but only K receivers, whereby the combiner chooses the K strongest signals from the N branches and then combines them using MRC. The SNR on the ith branch is γ_i which is iid distributed uniformly between 0 and 1 in linear units. Denote the SNR for the combiner output as γ_Σ; the outage probability for threshold γ_0 is $P_{out} = p(\gamma_\Sigma < \gamma_0)$. We will assume that there are $N = 3$ branches and the threshold is $\gamma_0 = 1$.

   (a) (5 pts) Suppose $K = 1$, state the form of diversity combining for this special case and compute P_{out}.

   (b) (5 pts) Repeat part (a) if $K = 3$.

   (c) (5 pts) Suppose $K = 2$. Show that $p(\gamma_{\max} < \gamma_0/2) \le P_{out} = p(\gamma_\Sigma < \gamma_0) \le p(\gamma_{\min} < \gamma_0/2)$ for $\gamma_{\min} = \min_i \gamma_i, \gamma_{\max} = \max_i \gamma_i$. Compute upper and lower bounds for P_{out}.

7. (10 pts) **Diversity gain computation**: In this problem we explore the concept of diversity gain under both outage probability and average probability of error metrics, and show they are the same for extreme SNR values. For this question consider M independent Rayleigh fading random variables, corresponding to the channel between a single transmit antenna and M receiver antennas. The average SNR at each antenna is given by $\mathtt{SNR}$. You may assume DPSK transmission.

   (a) (4 pts) Defining outage probability for $M > 1$ as the probability of the instantaneous SNR at all the receiver antennas being less than a constant c, compute $-\frac{\log(\mathtt{outage\ probability})}{\log(\mathtt{SNR})}$ as $\mathtt{SNR} \to \infty$. Does it depend on c ?

   (b) (4 pts) Now consider an alternative definition of diversity gain based on average error probability as

   $$-\frac{\log(\mathtt{average\ probability\ of\ error\ with\ MRC})}{\log(\mathtt{SNR})}.$$

   For $M > 1$, evaluate this as $\mathtt{SNR} \to \infty$.

   (c) (2 pts) Now evaluate the diversity gain according to both the definitions as $\mathtt{SNR} \to 0$. Are these definitions of diversity gain more suitable at high $\mathtt{SNR}$ or low $\mathtt{SNR}$ to describe the gains from having $M > 1$ independent fading paths?

8. (15 pts) **Order Statistics and SC Diversity**. This problem uses order statistics as an alternate method to derive the result of (7.11). For a set of n identically and independently distributed random variables, $X_1, X_2, \ldots, X_n$, the order statistics of these variables are given by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ where

   $$X_{(1)} = \min\left[X_1, X_2, \ldots, X_n\right] \qquad X_{(n)} = \max\left[X_1, X_2, \ldots, X_n\right].$$

   In general, $X_{(i)}$ denotes the ith largest value of $X_1, X_2, \ldots, X_n$, with $X_{(1)} \le X_{(2))} \le \ldots \le X_{(n)}$. A well-known result in order statistics states that if $X_1, \ldots X_n$ are i.i.d random variables drawn from an exponential distribution with mean $1/\lambda$, then the random variable $X_{(j)}$ has the same distribution as the function

$$X_{(j)} \stackrel{d}{=} \frac{1}{\lambda}\left(\sum_{i=1}^{j} \frac{Z_i}{n-i+1}\right), \tag{$*$}$$

where $Z_1, \ldots, Z_n$ are independent exponentially distributed random variables with mean 1.

Note: For two random variables A, B, the notation $A \stackrel{d}{=} B$ means that A and B are equal in distribution.

(a) (4 pts) For selection-combining (SC) of M i.i.d. branches under Rayleigh fading where each branch has the same average received power $\bar{\gamma}$, use the result of ($*$) to derive (7.11) in the reader.

(b) (3 pts) Suppose that $\bar{\gamma} = 20$ dB. Find P_{out} for the cases listed below assuming BPSK modulation with $P_b = 10^{-4}$. Note: it is not required to use ($*$) to answer this question. You may use the formulas for P_{out} provided in the reader.

   i. $M = 1$ (no diversity)

   ii. $M = 3$. Also provide an expression for P_{out} in terms of the CDF of $X_{(3)}$.

(c) (8 pts) Suppose that instead of an SC diversity scheme, we have a single transmitter sending to three users, each with a single antenna. Assume BPSK modulation is sent over each user's single-antenna channel and these channels experience i.i.d. Rayleigh fading with SNR $\bar{\gamma} = 20$ dB. A user is in outage if $P_b > 10^{-4}$. Compute the probability associated with each of the following events and state their relationship to the CDFs of the variables $X_{(1)}, X_{(2)}$, and $X_{(3)}$:

   i. No users are in outage.

   ii. One or fewer users are in outage.

   iii. Two or fewer users are in outage.

   iv. All users are in outage.