EE359 Discussion Session 5
Performance of Linear Modulation in Fading, Diversity

February 12, 2020

Announcements

- Midterm review on Wednesday, February 19, 4-6 pm, in Packard 364
- Midterm on Friday, February 21, 2-4 pm, in Hewlett 103
- OH hour changed for next week — see calendar!

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Today's Outline

- Moment Generating Functions
- Performance in Fading
- Diversity and Diversity Performance analysis

Moments of a Random Variable

Given a random variable $\mathbf{X}$, and its probability density function $f(x)$, its nth moment is given by:

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Why do we care? Gives useful characterization of random process

- $n = 1$ is the mean
- $n = 2$ is the variance
- Higher order moments often interesting

Moment Generating Function

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Intuition:

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = 1 + t\mathbb{E}\left[X\right] + \frac{t^2\mathbb{E}\left[X^2\right]}{2!} + \cdots$$

Differentiating and setting $t = 0$ gives moments!

Moment Generating Function

More concretely:

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Another useful observation:

$$\mathcal{L}\left\{f_X\right\}(s) = \int_{-\infty}^{\infty} e^{-sx} f_X(x) dx$$

$$M_X(t) = \mathcal{L}\left\{f_X\right\}(-t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

MGF can be found from two-sided Laplace tranform of PDF

Sums of Random Variables

MGFs are also useful for dealing with linear combinations of random variables

$$S_n = \sum_{i=1}^{n} a_i X_i,$$

- a_i arbitrary constants
- X_i independant (not necessarily identical) random variables.

PDF of S_n is found from convolution of each X_i.

Moments of S_n are given by:

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

Performance Metrics under Fading

System model

$$y[i] = \sqrt{\gamma[i]}x[i] + n[i]$$

Different metrics

- Average probability of error: Relevant when channel is fast fading
- Outage probability: Relevant when channel is slow fading
- Combined Outage + Avg. probability of error: shadowing (slow) and fading (fast)

Average Probability of Error

- Integrate the Q function over fading distributions
- Use change of integration order to try to get closed form expressions

Some useful relations

- P_b for BPSK in Rayleigh $\approx \frac{1}{4\bar{\gamma}}$ (Closed form also possible)
- P_b for DPSK in Rayleigh $\approx \frac{1}{2\bar{\gamma}}$ (Closed form possible)
- $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_0^{\pi/2} e^{-x^2/(2\sin^2\phi)} d\phi$

$\bar{P}_s$ using MGF ($\mathcal{M}_\gamma(s) = \int_0^\infty e^{s\gamma} p(\gamma) d\gamma$)

Idea

Use fact that

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \frac{1}{\pi} \int_0^{\pi/2} e^{-x^2/2\sin^2\phi} d\phi$$

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$$\begin{aligned}
\bar{P}_s &\approx \int_0^\infty \frac{\alpha_M}{\pi} Q(\sqrt{\beta_M \gamma}) p(\gamma) d\gamma \\
&= \frac{\alpha_M}{\pi} \int_{\gamma=0}^{\gamma=\infty} \int_{\phi=0}^{\phi=\pi/2} e^{-\beta_M \gamma/2\sin^2\phi} p(\gamma) d\phi d\gamma \\
&= \frac{\alpha_M}{\pi} \int_{\phi=0}^{\phi=\pi/2} \int_{\gamma=0}^{\gamma=\infty} e^{-\beta_M \gamma/2\sin^2\phi} p(\gamma) d\gamma d\phi \\
&= \frac{\alpha_M}{\pi} \int_{\phi=0}^{\phi=\pi/2} \mathcal{M}_\gamma(-\beta_M/2\sin^2\phi) d\phi
\end{aligned}$$

Example: BPSK in Rayleigh Fading

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$$M_{\gamma_b}(s) = (1 - s\bar{\gamma}_b)^{-1}$$
$$M_{\gamma_b}\left(-\frac{1}{\sin^2\phi}\right) = \left(1 + \frac{\bar{\gamma}_b}{\sin^2\phi}\right)^{-1}$$

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Integral now becomes:

$$\bar{P}_b = \frac{1}{\pi}\int_0^{\pi/2} \left(1 + \frac{\bar{\gamma}_b}{\sin^2\phi}\right)^{-1} d\phi$$

Error floors

As $\gamma_s \to \infty$, $P_{\text{error}} \to 0$ usually. Not true if there is an *error floor*!

Some reasons

- Differential modulation with large symbol times and/or fast fading (due to small T_c)
- Due to intersymbol interference ISI (or small B_c) $P_b \approx (\frac{\sigma}{T_s})^2$

Some factors

- Correlation function of channel (channel coherence time T_c and bandwidth B_c)
- Fading statistics, symbol time T_s

Question

What happens to error floors if T_s decreases or data rate increases?

Diversity

Idea

Use of independent fading realizations can reduce the probability of error/outage events

Some observations

- Diversity can be in time, space, frequency, polarization, ...
- Diversity order used as a measure of diversity, defined as

$$M = \lim_{\bar{\gamma}\to\infty} \frac{-\log P_e}{\log \bar{\gamma}}, \; P_e = \bar{P}_s \text{ or } P_{\text{out}}$$

- Can also use array gain (or SNR gain) $\bar{\gamma}_\Sigma / \bar{\gamma}$, where $\bar{\gamma}_\Sigma$ is the average SNR after "diversity combining"

Diversity order

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Specifying diversity order M is roughly equivalent to saying that at

$$\bar{P}_e \approx (\bar{\gamma})^{-M} \qquad \text{or} \qquad P_{\text{out}} \approx (P_{\text{out, 1 branch}})^M$$

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Array gain

Array gain A_g is equivalent to ratio of average SNRs after diversity combining

$$A_g = \frac{\bar{\gamma}_\Sigma}{\bar{\gamma}}$$

Diversity combining techniques

Two main schemes:

- Selection combining: Choose the largest SNR of the independent realizations
- Maximal ratio combining: Combine all the independent received SNRs to maximize SNR

Selection combining (SC)

Idea

Given M i.i.d. r.v., $\gamma_1, \dots, \gamma_M \geq 0$,

$$P(\max_i(\gamma_i) < c) = P(\gamma_i < c)^M$$

Some observations

- Define $\gamma_\Sigma = \max_i \gamma_i$
- In Rayleigh fading $\bar{\gamma}_\Sigma = \bar{\gamma}(\sum_{i=1}^M 1/i)$ ($\bar{\gamma}$: average SNR at a branch)
- $\bar{P}_b$ in general difficult, but for DPSK and Rayleigh fading,

$$\bar{P}_b = M/2 \sum_{m=0}^{M-1} (-1)^m \frac{\binom{M-1}{m}}{1 + m + \bar{\gamma}}$$

Selection combining continued

Outage probability

$$P_{\text{out}} = \left(1 - e^{-\frac{\gamma_0}{\bar{\gamma}}}\right)^M$$

Question (SC in Rayleigh fading)

- What is the diversity gain?:
- What is the SNR gain?:

Selection combining continued

Outage probability

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Question (SC in Rayleigh fading)

- What is the diversity gain?: M
- What is the SNR gain?: $\sum_i^M 1/i$

Maximal ratio combining (MRC)

Idea

Instead of discarding weaker branches, combine the SNRs of all branches, i.e.

$$\gamma_\Sigma = \sum_{i}^{M} \gamma_i$$

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Nuts and bolts

- Need to make the received components of the same phase (not a problem with modern DSP)
- Maximal ratio combining maximizes received SNR, i.e. solves the following problem

$$\max_{\mathbf{a}:||\mathbf{a}||^2=1} \frac{\mathsf{E}[|\mathbf{a}^H \boldsymbol{\gamma} x|^2]}{\mathsf{E}[|\mathbf{a}^H \mathbf{n}|^2]}$$

- MGF of sums decompose into product of individual MGFs so easy to analyse $\bar{P}_s$

MRC continued (Outage probability and $\bar{P}_s$)

Outage probability

$$P_{\text{out}} = 1 - e^{\frac{\gamma_0}{\bar{\gamma}}} \left(\sum_{i=0}^{M-1} \left(\frac{\gamma_0}{\bar{\gamma}} \right)^i / i! \right)$$

MRC continued (Outage probability and $\bar{P}_s$)

Outage probability

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Average probability of error $\bar{P}_s$

- The MGF of sum decouples into product of MGFs
- For DPSK and Rayleigh fading, average error probability is

$$\frac{1}{2} E_{\gamma_\Sigma}[e^{-\gamma_\Sigma}] = \frac{1}{2} \prod_{i=1}^{M} E_{\gamma_i}[e^{-\gamma_i}] = \frac{1}{2} \prod_{i=1}^{M} \mathcal{M}(-1)$$

- For general constellations

$$P_s = C \int_{\phi=A}^{\phi=B} (\mathcal{M}(-\gamma/2 \sin^2 \phi))^M d\phi$$

Questions

- What is the diversity order for MRC?:

- What is the SNR gain for MRC?:

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Note: Assumes perfect RX CSI